A two–parametric family of asymmetric exclusion processes
and its exact solution

M. Alimohammadi[a] [1]**, V. Karimipour**[b,c] [2] **M. Khorrami**[c,d] [3]

[a] *Department of Physics, University of Teheran, North Karegar,*
Tehran, Iran
[b] *Department of Physics, Sharif University of Technology,*
P.O.Box 11365-9161, Tehran, Iran
[c] *Institute for Studies in Theoretical Physics and Mathematics,*
P.O.Box 19395-5746, Tehran, Iran
[d] *Institute for Advanced Studies in Basic Sciences,*
P.O.Box 159,Gava Zang,Zanjan 45195,Iran.

Abstract

A two–parameter family of asymmetric exclusion processes for particles on a one-dimensional lattice is defined. The two parameters of the model control the driving force and an effect which we call pushing, due to the fact that particles can push each other in this model. We show that this model is exactly solvable via the coordinate Bethe Ansatz and show that its *N*-particle *S*-matrix is factorizable. We also study the interplay of the above effects in determining various steady state and dynamical characteristics of the system.

PACS numbers: 82.20.Mj, 02.50.Ga, 05.40.+j

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[1]e-mail:alimohmd@netware2.ipm.ac.ir

[2]e-mail:vahid@netware2.ipm.ac.ir

[3]e-mail:mamwad@netware2.ipm.ac.ir

1   Introduction

The asymmetric simple exclusion process (ASEP) is the simplest and most studied model of interacting particle systems in one dimension (see [1-5] and references therein). This model can be related via suitable mappings to many different physical models ranging from interface growth [6,7] to problems of traffic flow [8-11]. In this paper, however we would like to look at it only as a model for a collection of random walkers interacting with each other by simple exclusion. Each particle hops with rate $R(L)$ to it's right (left) neighboring site if this site is empty, otherwise it stops. When the hoping rates R and L are equal, one deals with the symmetric exclusion process. This case, models diffusion of particles on a one dimensional line in the absence of a driving force. The other extreme case (i.e. $L=0$), is usually called the totally asymmetric exclusion process (TASEP).

The asymmetric exclusion process has been extensively studied in the past few years as a prototype of a one dimensional system far from equilibrium for which some exact results can be obtained. Among the many aspects of this problem which have been studied, one can mention the mean field solution of the steady state and the phase structure [12,13], the exact steady state of the open [14], and closed chain [15], the effect of impurities [16-18], exact calculation of some dynamical properties [19],the effect of different kinds of updatings [20-21], and finally the exact calculation of conditional probabilities via the coordinate Bethe Ansatz [22].

In ref.[23] we considered the TASEP and added a new element into this process, namely the possibility that a particle pushes the particles in front of it with a rate depending on the number of these particles. Being interested in exactly solvable models, we found that if a particle can push a collection of n-adjacent particles in its immediate neighborhood with a rate given by

$$r_n = \frac{1}{1 + \frac{\lambda}{\mu} + \cdots + \left(\frac{\lambda}{\mu}\right)^n},$$

(1)

where $0 \leq \mu = 1 - \lambda \leq 1$, then the problem allows an exact solution via the coordinate Bethe Ansatz. The fact that r_n decreases with n is physically natural, although its functional form may seem peculiar. This particular form is dictated by our demand of the exact solvability of the model.Varying the parameter μ from 0 to 1, one can then smoothly interpolates between the TASEP and the so called drop-push model [24].

Remark: Here, by an exact solution we mean an exact determination of the time dependence of n-particle conditional probabilities on an infinite lattice. We also show that the steady state of the system on a ring is one in which all of the configurations have equal weights. The steady state of the system on an open chain is not known at present. Incidentally, the technique of matrix product ansatz does not work for this latter problem since the Hamiltonian of the process is highly nonlocal.

In the present work we consider the partially asymmetric case and study the combined effect of pushing and driving. Again we demand exact solvability and follow the strategy of our previous work [23]. The basic objects we are interested in are the probabilities $P(x_1, x_2, \cdots, x_N; t | y_1, y_2, \cdots, y_N)$ for finding at time t, particle 1 at point x_1, particle 2 at point $x_2, \cdots$, given their initial positions at points $y_1, y_2, \cdots$, respectively. For notational convenience, in the following we suppress the initial coordinates whenever no confusion can arise. The physical region for the coordinates is $x_1 < x_2 < ... < x_N$. If one wants to take into account a type of pushing effect, one should write a large number of equations depending on which and how many of the particles are adjacent to each other. The number of cases and hence the number of equations grows rapidly as the number of particles increases. The basic idea is to write one single equation supplemented by a particular boundary condition and to see what kind of pushing effect emerges. The master equation is

$$
\begin{aligned}
\frac{\partial}{\partial t} P(x_1, x_2, \cdots, x_N; t) &= R[P(x_1 - 1, x_2, \cdots, x_N; t) + \cdots + P(x_1, x_2, \cdots, x_N - 1; t)] \\
&\quad + L[P(x_1 + 1, x_2, \cdots, x_N; t) + \cdots + P(x_1, x_2, \cdots, x_N + 1; t)] \\
&\quad - NP(x_1, x_2, \cdots, x_N; t).
\end{aligned}
\tag{2}
$$

In the following we re-scale time so that $R + L = 1$. We assume that this equation holds in the whole physical region. When any of the coordinates of the left hand side are adjacent (say $x_{i+1} = x_i + 1$) , points on the boundary of the physical region appear on the right hand side of (2). We fix the value of these terms by the following boundary condition:

$$
P(x, x; t) = \lambda P(x, x + 1; t) + \mu P(x - 1, x; t),
\tag{3}
$$

where for simplicity we have suppressed all the other coordinates. This idea is most apparent in sectors of low particle number. For example, in the two particle sector, combination of (2) and (3) yields for adjacent particles

$$
\begin{aligned}
\frac{\partial}{\partial t} P(x, x + 1) &= R[P(x - 1, x + 1) + \mu P(x - 1, x) - (1 + \mu)P(x, x + 1)] \\
&\quad + L[P(x, x + 2) + \lambda P(x + 1, x + 2) - (1 + \lambda)P(x, x + 1)],
\end{aligned}
\tag{4}
$$

which means that a particle hops freely to the right and left with rates R and L and pushes its neighboring particle to the right and left with rates $R\mu$ and $L\lambda$, respectively. We will show that in the general case, equations (2) and (3) imply that the following processes :

$$
\underbrace{11 \cdots 1}_{n+1} \emptyset \longrightarrow \emptyset \underbrace{11 \cdots 1}_{n+1},
\tag{5}
$$

$$
\emptyset \underbrace{11 \cdots 1}_{n+1}, \longrightarrow \underbrace{11 \cdots 1}_{n+1} \emptyset
\tag{6}
$$

2

occur respectively with rates r_n and l_n given by

$$r_n = R\frac{1}{1 + \frac{\lambda}{\mu} + \cdots + \left(\frac{\lambda}{\mu}\right)^n}, \tag{7}$$

and

$$l_n = L\frac{1}{1 + \frac{\mu}{\lambda} + \cdots + \left(\frac{\mu}{\lambda}\right)^n}. \tag{8}$$

Remark: Interchanging particles and holes, this model can be seen to be equivalent to a model in which each particle does not only hops to its immediate neighboring sites but to any other vacant site, with rates depending on the hopping distance; i.e., in the equivalent model the following processes occur with rates r_n and l_n, respectively

$$\underbrace{\emptyset\emptyset\cdots\emptyset}_{n+1}1 \longrightarrow 1\underbrace{\emptyset\emptyset\cdots\emptyset}_{n+1} \tag{9}$$

$$1\underbrace{\emptyset\emptyset\cdots\emptyset}_{n+1} \longrightarrow \underbrace{\emptyset\emptyset\cdots\emptyset}_{n+1}1 \tag{10}$$

. The basic parameters of the model are $R = 1 - L$, and $\mu = \lambda - 1$. As we increase μ from 0 to 1, the effect of pushing to the right increases and that to the left decreases. For single particles the hopping rates to the right and left are still R and L respectively. Thus the driving is controlled by R or L and the pushing by μ or λ.

A few special cases are worth mentioning:

When $\mu = \lambda$, the pushing effect to the right and left are equal and the rates are

$$r_n = R\frac{1}{n+1} \qquad\qquad l_n = L\frac{1}{n+1}. \tag{11}$$

In this case, the asymmetry is controlled only by driving.

When $\mu = 1$, then we have maximum pushing to the right and no pushing to the left. The rates are

$$r_n = R, \qquad\qquad l_0 = L, \qquad l_{n>0} = 0$$

Single particles can hop to the left, but can not push other particles to the left.

When $L = 0$, particles hop to the right only, and by varying μ form 0 to 1 we interpolate between TASEP and the drop-push model [24]. Another way to understand the difference of the two sources of asymmetry in this problem is to note that the effect of driving can at least at long times be completely removed by going to an appropriate frame of reference while that of pushing can not. To see this, consider the transformation

$$x_i \rightarrow x_i' := x_i - Vt, \tag{12}$$

which is a Galilean boost. Actually, this is not an allowed transformation of our problem, since the x_i's are integers, whereas t is real. However, if the probability distribution is

sufficiently slowly varying (e.g. at long times), then one can define a probability density function with real variables. The master equation for this function is (to lowest order)

$$\frac{\partial P}{\partial t} = -(R - L) \sum_i \frac{\partial P}{\partial x_i} + \frac{1}{2} \sum_i \frac{\partial^2 P}{\partial x_i^2}. \tag{13}$$

In this case, the Galilean boost becomes a symmetry of the space–time being considered. Going to a reference frame moving with the velocity $V = R - L$, one obtains

$$\frac{\partial P}{\partial t} = \frac{1}{2} \sum_i \frac{\partial^2 P}{\partial x_i'^2}. \tag{14}$$

The asymmetry due to driving has been removed. However, the asymmetry due to pushing remains intact since the boundary condition does not change under this transformation.

The rest of this paper is devoted to the technical details and elaboration of the above results. In section 2 we prove that equations (2) and (3) actually give the above mentioned process. In section 3 we apply the coordinate Bethe ansatz and show that for this highly nonlocal process the N-particle S matrix is still factorizable. We also find the integral representation of the N-particle conditional probability distributions. In section 4 we show that in the two limiting cases $\mu = 1$ and $\lambda = 1$, the conditional probabilities can be expressed in closed form as $N \times N$ determinants. Section 5 is devoted to the mean field solution and discussion of the current density relation. In section 6 we calculate the drift and the diffusion rates for the two particle sector and compare our results with those of the ordinary partially ASEP [22]. Finally in section 7, we discuss the qualitative picture of the phases for open systems and also discuss the relation with ordinary ASEP in other updating schemes.

2 The Master Equation and the Process

The master equation is (2) supplemented by the boundary condition (3). For sectors of low number of particles (e.g. N=2,3), one can repeatedly use (3) to find the rates. For general sectors we use the following lemma.

Lemma: The boundary condition (3) implies

$$\begin{aligned}
P & (x, x + 1, \cdots, x + n - 1, x + n, x + n) = \\
& (1 - r_{n+1})P(x, x + 1, \cdots, x + n - 1, x + n, x + n + 1) + \\
& r_{n+1}P(x - 1, x, \cdots, x + n - 2, x + n - 1, x + n),
\end{aligned} \tag{15}$$

$$\begin{aligned}
P & (x, x, x + 1, \cdots, x + n - 1, x + n) = \\
& (1 - l_{n+1})P(x - 1, x, x + 1, \cdots, x + n - 1, x + n, x + n) + \\
& l_{n+1}P(x, x + 1, x + 2, \cdots, x + n - 1, x + n, x + n + 1),
\end{aligned} \tag{16}$$

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where r_n and l_n are given in (7) and (8).

The proof of this lemma is almost the same as that given in [23] and will not be repeated here. This lemma in fact states how to resolve the singularity in coordinates of a cluster of n adjacent particles with that of a single particle from right and left respectively.

Consider now the master equation for a collection of n adjacent particles:

$$
\frac{\partial}{\partial t}P(x, x+1, \cdots, x+n-1) =
$$
$$
R\left[\sum_{i=0}^{n-1} r_i P(x-1, x, \cdots, x+i-2, x+i-1, x+i+1, \cdots, x+n-1)\right.
$$
$$
- \left.\left(\sum_{i=0}^{n-1} r_i\right) P(x, x+1, \cdots, x+n-1)\right]
$$
$$
+ \ L\left[\sum_{i=0}^{n-1} l_{n-1-i} P(x, x+1, \cdots, x+i-1, x+i+1, x+i+2, \cdots, x+n)\right.
$$
$$
- \left.\left(\sum_{i=0}^{n-1} l_i\right) P(x, x+1, \cdots, x+n-1)\right]. \tag{17}
$$

It is now clear that the master equation (2) and the boundary condition (3) imply the processes (5)-(8).

3 The Bethe Ansatz Solution

3.1 The case of an infinite lattice

The Bethe–ansatz solution to the master equation (2) is

$$
P(x_1, \cdots, x_N; t) = e^{Et}\Psi(x_1, \cdots, x_N), \tag{18}
$$

where

$$
\Psi(x_1, \cdots, x_N) = \sum_\sigma A_\sigma e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}. \tag{19}
$$

Here $\mathbf{x}$ and $\mathbf{p}$ denote n-tuples of coordinates and momenta, respectively, the summation runs over the elements of the permutation group, and A_σ's are coefficients to be determined from the boundary condition (3). Inserting (18) in (2), we have

$$
R\sum_{j=1}^{N}\Psi(x_1, \cdots, x_j-1, \cdots, x_N) + L\sum_{j=1}^{N}\Psi(x_1, \cdots, x_j+1, \cdots, x_N) = (N+E)\Psi(x_1, \cdots, x_N), \tag{20}
$$

or

$$
\sum_\sigma A_\sigma e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}\left[R\sum_j e^{-i\sigma(p_j)} + L\sum_j e^{i\sigma(p_j)}\right] = (N+E)\Psi(x_1, \cdots, x_N). \tag{21}
$$

From this, one obtains (as one can remove σ from the summations in the parenthesis)

$$E = \sum_j E(p_j), \tag{22}$$

where

$$E(p_j) = Re^{-ip_j} + Le^{ip_j} - 1. \tag{23}$$

The next step is to determine the coefficients A_σ, so that the eigenfunction Ψ satisfies the boundary condition (3). It is seen that nothing from the master equation enters this boundary condition. So the solution to this boundary condition is just what was found in [23], that is

$$A_{\sigma\sigma_i} = S(\sigma(p_i), \sigma(p_{i+1}))A_\sigma, \tag{24}$$

where σ is an arbitrary element of the permutation group, and σ_i is the generator which only interchanges p_i and p_{i+1}. The elements of the two–particle scattering matrix,

$$S_{jk} := S(p_j, p_k) = -\frac{1 - \lambda e^{ip_k} - \mu e^{-ip_j}}{1 - \lambda e^{ip_j} - \mu e^{-ip_k}}, \tag{25}$$

are thus sufficient to calculate the scattering matrix in the general N-particle sector, and the latter factorizes in terms of the former. As the scattering matrix is just that obtained in [23], the same reasoning shows that there are no bound states. So we can write the conditional probability as

$$P(\mathbf{x}; t|\mathbf{y}; 0) = \int \prod_{j=1}^N \frac{\mathrm{d}p_j}{2\pi} e^{E(\mathbf{p})t - i\mathbf{p}\cdot\mathbf{y}} \Psi(\mathbf{x}, \mathbf{p}), \tag{26}$$

where the normalization of Ψ is chosen so that the coefficient of $e^{i\mathbf{P}\cdot\mathbf{x}}$ in Ψ is equal to unity. Integrations run over $0 \leq p_j \leq 2\pi$, and the poles of the scattering matrix are shifted from the integration region through the prescription

$$S(p_k, p_m) \to S(p_k + i\epsilon, p_m), \qquad k < m. \tag{27}$$

This ensures that in the physical region ($x_j < x_{j+1}$, $y_j < y_{j+1}$) we have

$$P(\mathbf{x}; 0|\mathbf{y}; 0) = \prod_j \delta_{x_j, y_j}. \tag{28}$$

Equation (26) is an integral representation for the conditional probabilities. More explicit information is obtained after calculating the integrals. For example in the two particle sector equation (26) is written as

$$
\begin{aligned}
P(x_1, x_2; t | y_1, y_2; 0) &= \int \frac{\mathrm{d}^2 p}{4\pi^2} e^{[E(p_1)+E(p_2)]t - i(p_1 y_1 + p_2 y_2)} \\
&\quad \times \left\{ e^{i(p_1 x_1 + p_2 x_2)} - \frac{1 - \lambda e^{ip_2} - \mu e^{-ip_1}}{1 + \epsilon - \lambda e^{ip_1} - \mu e^{-ip_2}} e^{i(p_1 x_2 + p_2 x_1)} \right\}. \quad (29)
\end{aligned}
$$

Using the variables $\xi := e^{ip_1}$ and $\eta := e^{-ip_2}$, a contour integration yields

$$
\begin{aligned}
P(x_1, x_2; t | y_1, y_2; 0) &= e^{-2t} \sum_{m,n} \left\{ \frac{(Lt)^{m+n+y_2-x_2}(Rt)^{m+n+x_1-y_1}}{n!(n+y_2-x_2)!m!(m+x_1-y_1)!} \right. \\
&\quad - \sum_{p,q} \binom{p+q}{p} \lambda^p \mu^q \frac{(Lt)^{m+n+y_2-x_1+q}(Rt)^{m+n+x_2-y_1+p}}{n!(n+y_2-x_1+q)!m!(m+x_2-y_1+p)!} \\
&\quad \left. \times \left[1 - \frac{\lambda(n+y_2-x_1+q)}{Lt} - \frac{\mu(m+x_2-y_1+p)}{Rt} \right] \right\}. \quad (30)
\end{aligned}
$$

Here all of the summations run from zero to infinity. A simple calculation shows that in the limit $L = 0$, the result of [23] is obtained.

3.2   The case of a periodic lattice

On an infinite lattice, the set of momenta of the eigenfunctions of the Hamiltonian are continuous. On a finite lattice, however, this set is discrete. To obtain this set, consider a lattice of M sites, on which N particles live. Now, another boundary condition should be added [25]

$$
\Psi(x_1, x_2, \cdots, x_N) = \Psi(x_2, x_3, \cdots, x_N, x_1 + M). \quad (31)
$$

This means that one cannot unambiguously define the *first* particle: one can interpret the *first* particle as the *last* one, provided its coordinate is enhanced by M, the period of the lattice. Applying the boundary condition (31) on the eigenfunction (19), we have

$$
\begin{aligned}
\sum_\sigma A_\sigma e^{i[\sigma(p_1)x_1 + \sigma(p_2)x_2 + \cdots + \sigma(p_N)x_N]} &= \sum_\sigma A_\sigma e^{i[\sigma(p_1)x_2 + \sigma(p_2)x_3 + \cdots + \sigma(p_N)(x_1+M)]} \\
&= \sum_\sigma A_{\sigma\sigma_0} e^{i[\sigma(p_2)x_2 + \sigma(p_3)x_3 + \cdots + \sigma(p_1)(x_1+M)]}, \quad (32)
\end{aligned}
$$

where

$$
\sigma_0(p_1, p_2, \cdots, p_N) := (p_2, p_3, \cdots, p_N, p_1). \quad (33)
$$

This yields

$$
A_\sigma = A_{\sigma\sigma_0} e^{iM\sigma(p_1)}. \quad (34)
$$

But,

$$
\sigma_0 = \sigma_1 \cdots \sigma_{N-1}. \quad (35)
$$

So, using (24),

$$
\begin{aligned}
A_{\sigma\sigma_0} &= A_{\sigma\sigma_1\cdots\sigma_{N-1}} \\
&= A_\sigma S[\sigma\sigma_1\cdots\sigma_{N-2}(p_{N-1}),\sigma\sigma_1\cdots\sigma_{N-2}(p_N)]\cdots S[\sigma(p_1),\sigma(p_2)] \\
&= A_\sigma S[\sigma(p_1),\sigma(p_N)]S[\sigma(p_1),\sigma(p_{N-1})]\cdots S[\sigma(p_1),\sigma(p_2)].
\end{aligned}
\tag{36}
$$

Combining this with (34), we arrive at

$$
e^{-iM\sigma(p_1)} = S[\sigma(p_1),\sigma(p_N)]\cdots S[\sigma(p_1),\sigma(p_2)],
\tag{37}
$$

which can be written as

$$
e^{-iMp_k} = \prod_{j\neq k} S(p_k,p_j).
\tag{38}
$$

These are the Bethe equations of the system, the solution of which provides the allowed set of discrete momenta. Note that the driving parameter enters only the energy equation, as in (23), and the pushing parameter enters only the Bethe equations. Denoting e^{-ip_k} by z_k, eqs. (38) can be rewritten as:

$$
z_k^M = \prod_{j\neq k} \frac{\lambda z_j^{-1} + \mu z_k - 1}{1 - \lambda z_k^{-1} - \mu z_j}
\tag{39}
$$

This system of equations have a symmetry, namely it is invariant under $z \longrightarrow z^{-1}, \lambda \longrightarrow \mu$. This means that if the set $z^\alpha := \{z_k^\alpha\}$ are the quantized momenta for the (λ,μ) system, then the set $\omega^\alpha := \{(z_k^\alpha)^{-1}\}$ are the quantized momenta for the (μ,λ) system. Thus if we know the spectrum of the former system (see eqs.(22) and (23)):

$$
E_\alpha = \sum_j (Rz_j^{(\alpha)} + L(z_j^{(\alpha)})^{-1} - 1),
\tag{40}
$$

then the spectrum of the latter system is also known

$$
E'_\alpha = \sum_j (R(z_j^{(\alpha)})^{-1} + L(z_j^{(\alpha)}) - 1)
\tag{41}
$$

In particular this means that part of the analysis of the spectrum of ordinary ASEP ($L = \mu = 0$) which has been done by Gwa and Sphon [25], can be applied to the drop-push model [24].

4   Closed Form of the conditional probabilities in the limiting cases $\lambda = 1$ and $\mu = 1$

In the special cases $\lambda = 0, 1$, one can use a determinant ansatz for the conditional probabilities [22,23]:

$$
P(\mathbf{x};t|\mathbf{y};0) = e^{-Nt}\det[G(\mathbf{x};t|\mathbf{y};0)],
\tag{42}
$$

where G is an $N \times N$ matrix with elements

$$G_{ij}(\mathbf{x}; t | \mathbf{y}; 0) = g_{i-j}(x_i - y_j; t). \tag{43}$$

Inserting (42) in (2), one obtains

$$\frac{\partial}{\partial t} G_{ij}(x; t) = R\, G_{ij}(x - 1; t) + L\, G_{ij}(x + 1; t). \tag{44}$$

The equations obtained by the boundary condition (3) (for $\lambda = 0, 1$) are

$$\begin{cases} g_{k-1}(x; t) = g_{k-1}(x - 1; t) + \beta g_k(x; t), & \lambda = 0 \\ g_{k+1}(x; t) = g_{k+1}(x + 1; t) + \beta g_k(x; t), & \lambda = 1. \end{cases} \tag{45}$$

Writing (44) in the form

$$\dot{g}_k(x; t) = R\, g_k(x - 1; t) + L\, g_k(x + 1; t), \tag{46}$$

and introducing the z-transform

$$\tilde{g}_k(z, t) := \sum_x z^x g_k(x; t), \tag{47}$$

we obtain from (45) the following

$$\begin{cases} \tilde{g}_{k-1}(z, t) = \dfrac{\beta}{1 - z} \tilde{g}_k(z, t), & \lambda = 0 \\[2mm] \tilde{g}_{k+1}(z, t) = \dfrac{\beta}{1 - 1/z} \tilde{g}_k(z, t), & \lambda = 1 \end{cases} \tag{48}$$

while (46) yields

$$\tilde{g}_k(z, t) = e^{(Rz + L/z)t} \tilde{g}_k(z, 0). \tag{49}$$

We also note that $\tilde{g}_k(z, t)$ is the z-transform of the one–particle sector probability; so

$$\tilde{g}_0(z, 0) = \sum_x z^x \delta_{x,0} = 1. \tag{50}$$

Combining (48), (49), and (50), we arrive at

$$\tilde{g}_k(z, t) = \begin{cases} e^{(Rz + L/z)t} \left(\dfrac{\beta}{1 - z} \right)^{-k}, & \lambda = 0 \\[4mm] e^{(Rz + L/z)t} \left(\dfrac{\beta}{1 - 1/z} \right)^{k}, & \lambda = 1. \end{cases} \tag{51}$$

The parameter β drops from the determinant, so that one can set it equal to an arbitrary number; we set it equal to unity. From this, one obtains

$$g_k(x; t) = \sum_{\substack{m,n=0 \\ n-m+x \geq 0}}^{\infty} (-1)^{n-m+x} \binom{k}{n - m + x} \frac{(Rt)^m (Lt)^n}{m! n!}, \qquad \lambda = 0, \tag{52}$$

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and

$$g_k(x;t) = \sum_{\substack{m,n=0 \\ n-m-x \geq 0}}^{\infty} (-1)^{n-m-x} \binom{-k}{n-m-x} \frac{(Lt)^m (Rt)^n}{m!n!}, \qquad \lambda = 1. \tag{53}$$

Note that

$$g_k^{(\lambda=1,R,L)}(x;t) = g_{-k}^{(\lambda=0,L,R)}(-x;t), \tag{54}$$

which is a special case of the symmetry under reflection. By this, we mean that the system of equations (2) and (3) is invariant under the following transformations:

$$x_i \longrightarrow -x_{N+1-i}, \qquad R \longrightarrow L, \qquad \lambda \longrightarrow \mu$$

5   Steady state of the system on a Ring

In this section we consider a ring of N sites on which M particles are hopping. The steady state of this system is the one in which all the configurations have equal weights.Thus all the steady state probabilities $P(x_1, x_2, ...x_M)$ are equal to a constant. Stationarity of this measure is proved by noting that $P(x_1, x_2, ...x_M) = constant$, satisfies both the master equation (2) and the boundary condition (3). Uniqueness of the measure is ensured by connectivity of the process,i.e. the fact that every configuration can be reached from any other by a sequence of transitions [26]. In this state one can calculate all the correlation functions by simple combinatorics. If n_k is the random variable at site k which is 1 if it is occupied and 0 if it is vacant, then it is well known [15] that

$$< n_j >= \frac{M}{N}, \qquad < n_j n_k >= \frac{M(M-1)}{N(N-1)}, \qquad < n_j n_k n_l >= \frac{M(M-1)(M-2)}{N(N-1)(N-2)}, \qquad \text{etc.} \tag{55}$$

In the thermodynamic limit, when $M \to \infty$ and $N \to \infty$ with $M/N = \rho$, this steady state approaches the uncorrelated steady state given by the mean field solution. What we want to do in this section is to find the current density relation for such a steady state. In general one can find the equation for the rate of change of the average density at site k, either by going to the Hamiltonian formalism and using the equations $\frac{d}{dt} < n_k >=< [n_k, H] >$, or by just looking at the process and determining the various ways in which this density decreases or increases. We follow this second approach which is more transparent and intuitive. The rate of change of density of particles can also be written as a continuity equation,i.e. $\frac{d}{dt} < n_k >= J_{k-1} - J_k$, where J_k is the current through site k. This current is the algebraic sum of a positive and a negative current

$$J_k := RJ_k^+ - LJ_k^-, \tag{56}$$

where J_k^+ and J_k^- are due to the hopping of particles to the right and to the left respectively. Due to the pushing effect, both of these currents are non-local. The explicit expression of

J_k^+ and J_k^- are:

$$
\begin{aligned}
J_k^+ &= r_0 < n_k(1 - n_{k+1}) > + r_1 \Big(< n_{k-1}n_k(1 - n_{k+1}) > + < n_k n_{k+1}(1 - n_{k+2}) > \Big) \\
&+ r_2 \Big(< n_{k-2}n_{k-1}n_k(1 - n_{k+1}) > + < n_{k-1}n_k n_{k+1}(1 - n_{k+2}) > \\
&+ < n_k n_{k+1}n_{k+2}(1 - n_{k+3}) > \Big) + \cdots,
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
J_k^- &= l_0 < (1 - n_k)n_{k+1} > + l_1 \Big(< (1 - n_k)n_{k+1}n_{k+2} > + < (1 - n_{k-1})n_k n_{k+1} > \Big) \\
&+ l_2 \Big(< (1 - n_k)n_{k+1}n_{k+2}n_{k+3} > + < (1 - n_{k-1})n_k n_k n_{k+1}n_{k+2} > \\
&+ < (1 - n_{k-2}n_{k-1}n_k n_{k+1}) > \Big) + \cdots.
\end{aligned}
\tag{58}
$$

A typical term like $< n_{k-1}n_k(1 - n_{k+1}) >$ in the expression of J_k^+, is in fact the probability of the configuration 110 on sites $k-1, k$, and $k+1$, respectively. We know from (5) and (7) that this configuration changes with rate r_1 to 011 on the same sites, hence this term contributes to the current J_k^+. A similar interpretation is true for other term $< n_k n_{k+1}(1 - n_{k+2}) >$, etc. In the uncorrelated steady state, the above currents are calculated to be

$$
J^+ := \rho(1 - \rho) \sum_{n=0}^{\infty} \frac{(n + 1)\rho^n}{1 + \frac{\lambda}{\mu} + \cdots + \left(\frac{\lambda}{\mu}\right)^n},
\tag{59}
$$

and

$$
J^- := \rho(1 - \rho) \sum_{n=0}^{\infty} \frac{(n + 1)\rho^n}{1 + \frac{\mu}{\lambda} + \cdots + \left(\frac{\mu}{\lambda}\right)^n}.
\tag{60}
$$

from which one obtains:

$$
J^+ = \begin{cases}
\rho(1 - \rho)(1 + \dfrac{2\mu}{\lambda}\rho) + o(\frac{\mu}{\lambda})^2, & \dfrac{\mu}{\lambda} << 1, \\[2mm]
\rho, & \dfrac{\mu}{\lambda} = 1, \\[2mm]
\dfrac{\rho}{1-\rho}(1 + \dfrac{\lambda}{\mu}\rho(\rho - 2)) + o(\frac{\lambda}{\mu})^2, & \dfrac{\mu}{\lambda} >> 1,
\end{cases}
\tag{61}
$$

and

$$
J^- = \begin{cases}
\dfrac{\rho}{1-\rho}(1 + \dfrac{\mu}{\lambda}\rho(\rho - 2)) + o(\frac{\mu}{\lambda})^2, & \dfrac{\mu}{\lambda} << 1 \\[2mm]
\rho, & \dfrac{\mu}{\lambda} = 1, \\[2mm]
\rho(1 - \rho)(1 + \dfrac{2\lambda}{\mu}\rho) + o(\frac{\lambda}{\mu})^2,
\end{cases}
\tag{62}
$$

Consider J^+: It is seen that as far as the pushing effect to the right is small ($\frac{\mu}{\lambda} << 1$), the standard mean field current of the ASEP gets only corrections of the order $\frac{\mu}{\lambda}$. For medium pushing, when $\frac{\mu}{\lambda} \approx 1$, the current is exactly equal to the density and the interesting point is

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that contrary to the case of ASEP, even when the lattice is filled with particles ($\rho = 1$), there is a nonzero current due to pushing. At very strong pushing ($\frac{\mu}{\lambda} >> 1$), the current even diverges when the lattice is filled . It is now instructive to consider only the leading terms of the total current J_k in different regimes. In the steady state, this current is independent of the site number k and hence is denoted by J. We find from (61) and (62) the following

$$
J = \begin{cases} \rho\left(R(1-\rho) - \frac{L}{1-\rho}\right) + o(\frac{\mu}{\lambda}), & \frac{\mu}{\lambda} << 1, \\ (R-L)\rho, & \frac{\mu}{\lambda} = 1, \\ \rho\left(\frac{R}{1-\rho} - L(1-\rho)\right) + o(\frac{\lambda}{\mu}), & \frac{\mu}{\lambda} >> 1. \end{cases} \tag{63}
$$

Remark: Note that our model incorporates only the totally ASEP as a special case . Therefore when $L \neq 0$, the above results should not coincide with those of the partially ASEP in the limiting case $\mu = 0$.

Consider the case $\frac{\mu}{\lambda} << 1$, where we have strong pushing to the left and weak pushing to the right. It is seen that for $L > R$ the current is always negative, which is expectable on physical grounds. However, for $L < R$, when the driving force is to the right, the two effects act in opposite directions. The current is positive as long as $\rho < \rho_c := 1 - \sqrt{\frac{L}{R}}$. At $\rho = \rho_c$ the current vanishes and for $\rho > \rho_c$, the pushing effect takes over and the current becomes negative.

6   Drift and Diffusion Rates in the Two Particle Sectors

In this section we want to study the interplay of driving and pushing in the behaviour of two important dynamical quantities, namely the drift and the diffusion rates. More generally, we study the long time behaviour of the quantities $\frac{d}{dt} < X >$ and $\frac{d}{dt}(< X^2 > - < X >^2)$. Our starting point is the exact calculation of conditional probability of the the two particles being a distance x apart, given their initial separation y. Denoted by $P_r(x; t|y; 0)$, it is given as:

$$
\begin{aligned}
P_r(x; t|y, ; 0) &= \sum_{x_2=-\infty}^{\infty} P(x_2 - x, x_2; t|0, y; 0) \\
&= \int \frac{\mathrm{d}^2 p}{4\pi^2} e^{Et - i\mathbf{p}\cdot\mathbf{y}} \\
&\quad \times \sum_{x_2} e^{i(p_1+p_2)x_2} \left[e^{-ip_1 x} + S(p_1, p_2) e^{-ip_2 x} \right] \\
&= \int \frac{\mathrm{d}p}{2\pi} e^{[E(p)+E(-p)]t+ip(y)} (e^{-ipx} + e^{-ip} e^{ipx}),
\end{aligned} \tag{64}
$$

where we have used:

$$
S(p, -p) = e^{-ip}. \tag{65}
$$

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From these we arrive at

$$P_r(x;t|y;0) = e^{-2t} \left[I_{y-x}(2t) + I_{y+x-1}(2t) \right], \tag{66}$$

where I_n is the modified Bessel function of order n with the integral representation

$$I_n(u) = \int \frac{dp}{2\pi} e^{inp+u\cos p}. \tag{67}$$

It is interesting to note that this probability is independent of the asymmetry- and drift-parameters. Another way to see this result, is to derive the equation of evolution for P_r. To do so, begin from the master equation (2) for two particles. Using the definition of $P_r(x;t|y;0)$ now abbreviated to $P_r(x)$, we arrive at

$$\dot{P}_r(x) = P_r(x-1) + P_r(x+1) - 2P_r(x). \tag{68}$$

The boundary condition (3) is transformed into

$$P_r(0) = P_r(1). \tag{69}$$

It is seen that the driving and the pushing parameters are absent in this equation. The physical explanation for the absence of μ or λ is that, when two particles push each other they do not change their inter-particle distance. This distance increases by one unit with rate $R + L$ (particle 2 hoping to the right or particle 1 to the left), and decreases by one unit with the same rate (particle 2 hoping to the left or particle 1 to the right), and since $R + L$ has been rescaled to unity, the driving parameters do not appear in these equations either. We should stress that this is not the case in more than two-particle sectors and the probabilities for relative distance in these sectors do indeed depend on the above parameters. The next quantities we calculate are the average velocities of particle 1 and particle 2. Note that particles can not overtake each other and that they keep their initial order at all times. We have:

$$< x_i >:= \sum_x x P_i(x) \tag{70}$$

where $P_i(x)$ is the probability of finding particle i at site x . The master equation for these probabilities are obtained from (2) and (3), using the definitions:

$$P_1(x) := \sum_{x_2=x+1}^{\infty} P(x,x_2), \tag{71}$$

and

$$P_2(x) := \sum_{x_1=-\infty}^{x-1} P(x_1,x). \tag{72}$$

This calculation finally leads to the following equations

$$\dot{P}_1(x) = R\{P_1(x-1) - P(x-1,x) + \mu P(x-1,x)\} - (x \longrightarrow x+1)$$

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$$+L\{P_1(x+1)+\lambda P(x+1,x+2)\}-(x \longrightarrow x-1), \tag{73}$$

and

$$
\begin{aligned}
\dot{P}_2(x) &= R\{P_2(x-1)+\mu P(x-2,x-1)\}-(x \longrightarrow x+1)\\
&\quad +L\{P_2(x+1)-P(x,x+1)+\lambda P(x,x+1)\}-(x \longrightarrow x-1).
\end{aligned} \tag{74}
$$

We have written these equations in this unsimplified form in order to convey their simple physical meaning. In fact they can also be obtained by intuitive reasoning . Consider for example eq. (73). The first two terms in the curly bracket of the first line are due to particle 1 at site $x-1$ hopping to an already vacant site at x and the third term is due to particle 1 at site $x-1$ hopping to site x and pushing the already present particle 2 at this site to the right. Other terms have similar meaning. One now obtains from (70) and (73,74) the following

$$\frac{\mathrm{d}<x_1>}{\mathrm{d}t}=R-L-\lambda P_r(1), \tag{75}$$

and

$$\frac{\mathrm{d}<x_2>}{\mathrm{d}t}=R-L+\mu P_r(1). \tag{76}$$

One can even derive these equations from the beginning by physical reasoning without using the master equation. For example we know that the hopping rate of particle 1 to the right is normally R unless it is one site behind particle 2 where its hopping rate will be $R\mu$. These two terms can be combined to give a positive contribution $R+(R\mu-R)P_r(1)$ to the average velocity of particle 1. On the other hand its hopping rate to the left is normally L unless particle 2 is exactly one site to its right, where its hopping rate becomes $L+L\lambda$. These two terms are then combined to give a negative contribution $-L-L\lambda P_r(1))$ to the average velocity. Adding these, one obtains (75). The same kind of reasoning can give (76). Using the definitions

$$<r>:=<x_2>-<x_1> \qquad\qquad <X>:=\frac{1}{2}(<x_1>+<x_2>) \tag{77}$$

we find

$$\frac{\mathrm{d}<r>}{\mathrm{d}t}=P_r(1), \tag{78}$$

and

$$
\begin{aligned}
\frac{\mathrm{d}<X>}{\mathrm{d}t} &= R-L+\frac{\mu-\lambda}{2}P_r(1)\\
&= R-L+\frac{\mu-\lambda}{2}\frac{\mathrm{d}<r>}{\mathrm{d}t}.
\end{aligned} \tag{79}
$$

From these, one obtains

$$<X>=<X>_0+(R-L)t+\frac{\mu-\lambda}{2}(<r>-<r>_0), \tag{80}$$

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where the subscript 0 refers to initial conditions. From (66), and using the asymptotic behaviour of the modified Bessel functions, one can obtain the asymptotic behaviour of these expectation values. We have

$$P_r(1) = \frac{1}{\sqrt{\pi t}} + O(t^{-3/2}). \tag{81}$$

Then, we obtain

$$< r > = C + 2\sqrt{\frac{t}{\pi}} + O(t^{-1/2}), \tag{82}$$

where C is a constant depending on the initial conditions. So,

$$< X > = < X >_0 + (R - L)t + \frac{\mu - \lambda}{2}\left(C + 2\sqrt{\frac{t}{\pi}} - < r >_0\right) + O(t^{-1/2}). \tag{83}$$

At long times we have:

$$\frac{\mathrm{d} < X >}{\mathrm{d}t} = R - L + \frac{\mu - \lambda}{2\sqrt{\pi t}} + O(t^{-3/2}). \tag{84}$$

It is seen that to leading order the drift rate defined as $V := \frac{d}{dt} < X >$ is only controlled by driving, and pushing has only a sub-leading effect. To calculate the diffusion rate we proceed as follows:

$$\frac{\mathrm{d}(< X^2 > - < X >^2)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} < x_1^2 > + \frac{1}{2} < x_2^2 >\right) - \frac{\mathrm{d}(< X >^2)}{\mathrm{d}t}, \tag{85}$$

where

$$< x_i^2 > = \sum_x x^2\, P_i(x), \qquad i = 1, 2. \tag{86}$$

Using (73) and (74), we arrive at

$$\frac{\mathrm{d} < x_1^2 >}{\mathrm{d}t} = 1 + 2(R - L) < x_1 > + \lambda(L - R)P_r(1) - 2\lambda\sum_x x\, P(x, x + 1), \tag{87}$$

and

$$\frac{\mathrm{d} < x_2^2 >}{\mathrm{d}t} = 1 + 2(R - L) < x_2 > + \mu(R - L)P_r(1) + 2\mu\sum_x x\, P(x, x + 1). \tag{88}$$

From these we obtain

$$\begin{aligned}\frac{\mathrm{d}(< X^2 > - < X >^2)}{\mathrm{d}t} &= 1 - (\mu - \lambda) < X > P_r(1) + \frac{1 + (\mu - \lambda)(R - L)}{2}P_r(1) \\ &+ \frac{\mu - \lambda}{2}\sum_x(2x - 1)P(x - 1, x)\end{aligned} \tag{89}$$

The value of the quantity $\sum_x(2x - 1)P(x - 1, x)$ is calculated in the appendix . Inserting its value in the above formula, using the asymptotic form of the modified Bessel function, one obtains

$$\Delta := \lim_{t \to \infty}\frac{\mathrm{d}(< X^2 > - < X >^2)}{\mathrm{d}t} = 1 + (\lambda - \mu)^2\left(\frac{1}{2} - \frac{1}{\pi}\right). \tag{90}$$

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We see that pushing has a leading effect on the diffusion rate. The physical explanation behind this is that although the average distance between the particles grows with time as $t^{\frac{1}{2}}$ (see (82)), the width of the probability distribution (wave packet) of each particle increases also with rate $t^{\frac{1}{2}}$, thus the wave packets always overlap and there is always a finite probability that the particles push each other.

7   Discussion and Summary

We have defined a generalized asymmetric exclusion process with random sequential updating in which particles besides hoping randomly to the left and right can also push their neighboring particles with rates depending on the number of these particles. We have shown that this model, although governed by a very non-local Hamiltonian of the spin chain type, is exactly solvable via the coordinate Bethe ansatz. (For the type of the Hamiltonian see [23]). Throughout the paper we have tried to study the interplay between the two sources of asymmetry, the one due to driving and the one due to pushing. Due to the pushing effect, in our model, cluster of particles can also hop to right and left. This is similar to what happens in sequential updating schemes [20,21], where in one complete updating of the lattice, clusters of particles move. However, we remark that here we have this effect in continuous time and not discrete time. Specially, in sequential updating scheme, when the total updating operator is the product of local updating operators, the probability of hopping of clusters in one complete update, turns out to be a power of the hopping rates of single particles while in our model this is not so. That is why our Hamiltonian is very non-local.

The fact that despite this non-locality the system has an exact solution and its S-matrix is factorizable is interesting. What remains to be done for this model is to study its steady state (particularly in the totally asymmetric case $L=0$) and its phase structure on open systems when particles are injected and extracted at the open ends, to see how the simple phase diagram of the ASEP will be modified due to pushing. Again, due to the non-locality of the Hamiltonian of the process, the conventional technique of Matrix Product Ansatz can not be applied to this problem. Qualitative pictures may be obtained along the work of ref. [27]. However, one should first decide as to how to add boundary terms to this process, that is if particle arrive only at the already vacant boundary site or else, they can also push a cluster of particles of arbitrary size, already present there. If this is so, then there will be no longer a genuine difference between the boundary terms and the bulk. We believe that, due to these complications and the nonlocal character of the process, this problem deserves a separate study.

8 Appendix

To calculate the quantity $\sum_x (2x-1)P(x-1,x)$ needed in section 6, we proceed as follows:

$$
\begin{aligned}
\sum_x (2x-1)P(x-1,x) &= \int \frac{\mathrm{d}^2 p}{4\pi^2 i} e^{Et - i\mathbf{p}\cdot\mathbf{y}} \left[\left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} \right) \sum_x e^{i(p_1+p_2)x - ip_1} \right. \\
&\quad \left. + S_{12} \left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} \right) \sum_x e^{i(p_1+p_2)x - ip_2} \right] \\
&= \int \frac{i\mathrm{d}^2 p}{2\pi} \delta(p_1+p_2) \left[e^{-ip_1} \left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} \right) e^{Et - i\mathbf{p}\cdot\mathbf{y}} \right. \\
&\quad \left. + e^{-ip_2} \left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} \right) \left(S_{12} e^{Et - i\mathbf{p}\cdot\mathbf{y}} \right) \right] \\
&= e^{-2t} \Big\{ (y_1+y_2)[\mathrm{I}_y(2t) + \mathrm{I}_{y-1}(2t)] \\
&\quad + t(R-L)[\mathrm{I}_{y+1}(2t) + \mathrm{I}_y(2t) + \mathrm{I}_{y-1}(2t) + \mathrm{I}_{y-2}(2t)] \\
&\quad + (\mu - \lambda) \sum_{k=0}^{\infty} [\mathrm{I}_{y+k}(2t) + \mathrm{I}_{y+k+1}(2t)] \Big\}.
\end{aligned}
\tag{91}
$$

Using the identity

$$
\sum_{n=-\infty}^{\infty} I_n(2t) = e^{2t},
\tag{92}
$$

we arrive at

$$
\begin{aligned}
\sum_x (2x-1)P(x-1,x) &= e^{-2t} \{ (y_1+y_2)[\mathrm{I}_y(2t) + \mathrm{I}_{y-1}(2t)] \\
&\quad + t(R-L)[\mathrm{I}_{y+1}(2t) + \mathrm{I}_y(2t) + \mathrm{I}_{y-1}(2t) + \mathrm{I}_{y-2}(2t)] \} \\
&\quad + (\mu - \lambda) \Big\{ 1 - e^{-2t} \sum_{n=0}^{y-1} [\mathrm{I}_n(2t) + \mathrm{I}_{n+1}(2t)] \Big\}.
\end{aligned}
\tag{93}
$$

References

[1] T. M. Liggett, " *Interacting Particle Systems* " (Springer, 1985).

[2] H. Spohn, " *Large Scale Dynamics of Interacting Particles* " (Springer, 1991).

[3] D. Dhar, Phase Transition **9**,51 (1987).

[4] B. Derrida, Phys. Rep. **301**,65 (1998).

[5] B. Derrida and M. R. Evans, in " *Non-Equilibrium Statistical Mechanics in one Dimension* ", V. Privman ed. (Cambridge University Press, 1997), pp. 277.

[6] T. Halpin-Healy and Y. C. Zhang, Phys. Rep. **254**,215 (1995).

[7] J. Krug and H. Spohn, in " *Solids far from equilibrium* ", ed. C. Godreche (Cambridge, Cambridge University Press, 1991) and references therein.

[8] M. Schreckenberg, A. Schadschneider, K. Nagel and N. Ito, Phys. Rev. E **51**,2339 (1995).

[9] O. Biham, A.A.Middelton and D. Levine, Phys. Rev. A**46**,6124 (1992).

[10] K. Nagel, Phys. Rev. E**53**,4655 (1996).

[11] F. Spitzer, Adv. Math. **5**,246 (1970).

[12] G. Schütz and E. Domany, J. Stat. Phys. **72**,277 (1993).

[13] B. Derrida, E. Domany and D. Mukamel, J. Stat. Phys. **69**,667 (1992).

[14] B. Derrida, M.R. Evans, V. Hakim and V. Pasquier, J. Phys. A: Math. Gen. **26**,1493 (1993).

[15] P. Meakin, P. Ramanlal, L. Sander and R. C. Ball, Phys. Rev. A**34**,5091 (1986).

[16] S. A. Janowsky and J. L. Lebowitz, J. Stat. Phys. **77**,35 (1994), Phys. Rev. A**45**,618 (1992).

[17] B. Derrida, S. A. Janowsky, J. L. Lebowitz, and E. R. Speer, Europhys. Lett. **22**,651 (1993), J. Stat. Phys. **73**,813 (1993).

[18] H.W. Lee V. Popkov and D. Kim, J. Phys. A: Math. Gen. **30**,8497 (1997).

[19] B. Derrida and K. Mallick, J. Phys. A: Math. Gen. **30**,1031 (1997),
B. Derrida, M.R. Evans and K. Mallick, J. Stat. Phys. **79**,833 (1995).

[20] N. Rajewsky and M. Schreckenberg, Physica A**245**,139 (1997),
N. Rajewsky, L. Santen, A. Schadschneider, and M. Schreckenberg, J. Stat. Phys. **92**,151 (1998).

[21] H. Hinrichsen, J. Phys. A: Math. Gen. **29**,3659 (1996).

[22] G. M. Schütz, J. Stat. Phys. **88**,427 (1997).

[23] M. Alimohammadi, V. Karimipour, and M. Khorrami, Phys. Rev. E**57**,6370 (1998).

[24] G. M. Schütz, R. Ramaswamy, and M. Barma, J. Phys. A: Math. Gen. **29**,837 (1996).

[25] L. H. Gwa and H. Spohn, Phys. Rev. A**46**,844 (1992).

[26] N. G. van Kampen, *" Stochastic processes in physics and chemistry "* (North-Holland, Amsterdam, 1981).

[27] A. B. Kolomeisky, G. M. Schütz, E. B. Kolomeisky, and J. P. Straley, J. Phys. A: Math. Gen. **31**,6911 (1998).